

ON A MINIMAL SET OF GENERATORS FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES AS A MODULE OVER THE STEENROD ALGEBRA

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ABSTRACT. Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \dots, x_k]$ over the prime field of two elements, \mathbb{F}_2 , with the degree of each x_i being 1. We study the *hit problem*, set up by Frank Peterson, of determining a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, \mathcal{A} . In this paper, we explicitly determine a minimal set of \mathcal{A} -generators for P_k in the case $k = 5$ and the degree $4(2^d - 1)$ with d an arbitrary positive integer.

1. INTRODUCTION

Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \dots, x_k]$, with the degree of each x_i being 1. This algebra arises as the cohomology with coefficients in \mathbb{F}_2 of an elementary abelian 2-group of rank k . Then, P_k is a module over the mod-2 Steenrod algebra, \mathcal{A} . The action of \mathcal{A} on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula (see Steenrod and Epstein [19]).

An element g in P_k is called *hit* if it belongs to \mathcal{A}^+P_k , where \mathcal{A}^+ is the augmentation ideal of \mathcal{A} . That means g can be written as a finite sum $g = \sum_{u \geq 0} Sq^{2^u}(g_u)$ for suitable polynomials $g_u \in P_k$.

We study the *Peterson hit problem* of determining a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to determine a basis of the \mathbb{F}_2 -vector space $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. The problem is an interesting and important one. It was first studied by Peterson [11], Wood [24], Singer [17], and Priddy [13], who showed its relation to several classical problems respectively in cobordism theory, modular representation theory, the Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. Then, this problem was investigated by many authors (see Boardman [1], Bruner, Hà and Hung [2], Carlisle and Wood [3], Crabb and Hubbuck [4], Hưng [5], Hưng and Nam [6], Janfada and Wood [7], Kameko [8], Mothebe [9], Nam [10], Repka and Selick [14], Silverman [15], Silverman and Singer [16], Singer [18], Walker and Wood [22, 23], Wood [25], the second named author [20, 21] and others).

From the results of Wood [24] and Kameko [8], the hit problem is reduced to the case of degree n of the form

$$n = s(2^d - 1) + 2^d m, \quad (1.1)$$

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where s, d, m are non-negative integers and $1 \leq s < k$ (see [21].) For $s = k - 1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [4], Nam [10], Repka and Selick [14] and the second named author [20, 21].

In the present paper, we explicitly determine the hit problem in degree n of the form (1.1) with $s = k - 1 = 4$, $m = 0$ and d an arbitrary positive integer. The main result of the paper is the following.

Main Theorem. *Let $n = 4(2^d - 1)$ with d a positive integer. The dimension of the \mathbb{F}_2 -vector space $(QP_5)_n$ is determined by the following table:*

$n = 4(2^d - 1)$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d \geq 5$
$\dim(QP_5)_n$	45	190	480	650	651

In Section 2, we recall some needed information on the admissible monomials in P_k , Singer's criterion on the hit monomials and Kameko's homomorphism. The proof of Main Theorem is presented in Section 3.

2. PRELIMINARIES

In this section, we recall some needed information from Kameko [8], Singer [18] and the second named author [21] which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \dots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \dots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \dots, j_s\} \subset \mathbb{N}_k,$$

In particular, we set

$$\begin{aligned} X_{\mathbb{N}_k} &= 1, \\ X_{\emptyset} &= x_1 x_2 \dots x_k, \\ X_j &= x_1 \dots \hat{x}_j \dots x_k, \quad 1 \leq j \leq k, \end{aligned}$$

and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i -th coefficient in dyadic expansion of a non-negative integer a . That means

$$a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \dots,$$

for $\alpha_i(a) = 0$ or 1 with $i \geq 0$.

Let $x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j$, $1 \leq j \leq k$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geq 0$. Then, we have $x = \prod_{t \geq 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. For a monomial $x \in P_k$, define two sequences associated with x by

$$\begin{aligned} \omega(x) &= (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots), \\ \sigma(x) &= (\nu_1(x), \nu_2(x), \dots, \nu_k(x)), \end{aligned}$$

where $\omega_i(x) = \sum_{1 \leq j \leq k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geq 1$. The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of x .

Let $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector ω , we define $\deg \omega = \sum_{i>0} 2^{i-1} \omega_i$. If there are $i_0 = 0, i_1, i_2, \dots, i_r > 0$ such that

$$\begin{aligned} i_1 + i_2 + \dots + i_r &= m, \\ \omega_{i_0 + \dots + i_{s-1} + t} &= b_s, 1 \leq t \leq i_s, 1 \leq s \leq r, \end{aligned}$$

and $\omega_i = 0$ for all $i > m$, then we write $\omega = (b_1^{(i_1)}, b_2^{(i_2)}, \dots, b_r^{(i_r)})$. Denote $b_u^{(1)} = b_u$. For example, $\omega = (3, 3, 2, 2, 2, 1, 0, \dots) = (3^{(2)}, 2^{(3)}, 1)$.

Denote by $P_k(\omega)$ the subspace of P_k spanned by all monomials y such that $\deg y = \deg \omega$, $\omega(y) \leq \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k .

- i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+ P_k$. If $f \equiv 0$, then f is called *hit*.
- ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+ P_k + P_k^-(\omega)$.

Obviously, the relations \equiv and \equiv_ω are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation \equiv_ω . Then, we have

$$QP_k(\omega) = P_k(\omega) / ((\mathcal{A}^+ P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the class in QP_k represented by f . If ω is a weight vector and $f \in P_k(\omega)$, then denote by $[f]_\omega$ the class in $QP_k(\omega)$ represented by f . Denote by $|S|$ the cardinal of a set S .

It is easy to see that

$$QP_k(\omega) \cong QP_k^\omega := \langle \{[x] \in QP_k : x \text{ is admissible and } \omega(x) = \omega\} \rangle.$$

So, we get

$$(QP_k)_n = \bigoplus_{\deg \omega = n} QP_k^\omega \cong \bigoplus_{\deg \omega = n} QP_k(\omega).$$

Hence, we can identify the vector space $QP_k(\omega)$ with $QP_k^\omega \subset QP_k$.

Definition 2.4. Let x, y be monomials of the same degree in P_k . We say that $x < y$ if and only if one of the following holds:

- i) $\omega(x) < \omega(y)$;
- ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.5. A monomial x in P_k is said to be inadmissible if there exist monomials y_1, y_2, \dots, y_t such that $y_j < x$ for $j = 1, 2, \dots, t$ and $x - \sum_{j=1}^t y_j \in \mathcal{A}^+ P_k$. A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of \mathcal{A} -generators for P_k in degree n .

Definition 2.6. A monomial x in P_k is said to be strictly inadmissible if and only if there exist monomials y_1, y_2, \dots, y_t such that $y_j < x$, for $j = 1, 2, \dots, t$ and

$$x = \sum_{j=1}^t y_j + \sum_{u=1}^{2^s-1} S q^u(h_u)$$

with $s = \max\{i : \omega_i(x) > 0\}$ and suitable polynomials $h_u \in P_k$.

It is easy to see that if x is strictly inadmissible, then it is inadmissible.

Theorem 2.7. (Kameko [8], Sum [20]) *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

- i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*
- ii) *If w is strictly inadmissible, then wy^{2^s} is also strictly inadmissible.*

Now, we recall a result of Singer [18] on the hit monomials in P_k .

Definition 2.8. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \dots, k$. If z is a spike with $d_1 > d_2 > \dots > d_{r-1} \geq d_r > 0$ and $d_j = 0$ for $j > r$, then it is called the minimal spike.

For a positive integer n , by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1 \leq i \leq r} (2^{d_i} - 1)$, where $d_i > 0$. In [18], Singer showed that if $\mu(n) \leq k$, then there exists uniquely a minimal spike of degree n in P_k . The following is a criterion for the hit monomials in P_k .

Theorem 2.9. (Singer [18]) *Suppose $x \in P_k$ is a monomial of degree n , where $\mu(n) \leq k$. Let z be the minimal spike of degree n . If $\omega(x) < \omega(z)$, then x is hit.*

This result implies the one of Wood, which originally is a conjecture of Peterson [11].

Theorem 2.10. (Wood [24]) *If $\mu(n) > k$, then $(QP_k)_n = 0$.*

One of the main tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}_*^0 : QP_k \rightarrow QP_k$. This homomorphism is induced by the \mathbb{F}_2 -linear map, also denoted by $\widetilde{Sq}_*^0 : P_k \rightarrow P_k$, given by

$$\widetilde{Sq}_*^0(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \dots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. Note that \widetilde{Sq}_*^0 is not an \mathcal{A} -homomorphism. However, $\widetilde{Sq}_*^0 Sq^{2t} = Sq^t \widetilde{Sq}_*^0$, and $\widetilde{Sq}_*^0 Sq^{2t+1} = 0$ for any non-negative integer t .

Denote by $(\widetilde{Sq}_*^0)_{(k,m)} : (QP_k)_{2m+k} \rightarrow (QP_k)_m$ Kameko's homomorphism in degree $2m + k$.

Theorem 2.11. (Kameko [8]) *Let m be a positive integer. If $\mu(2m + k) = k$, then*

$$(\widetilde{Sq}_*^0)_{(k,m)} : (QP_k)_{2m+k} \rightarrow (QP_k)_m$$

is an isomorphism of the \mathbb{F}_2 -vector spaces.

By combining Theorems 2.10 and 2.11, one can see that the hit problem is reduced to the case of degree n of the form (1.1) as given in the introduction.

We set

$$\begin{aligned} P_k^0 &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_1 a_2 \dots a_k = 0\} \rangle, \\ P_k^+ &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_1 a_2 \dots a_k > 0\} \rangle. \end{aligned}$$

It is easy to see that P_k^0 and P_k^+ are the \mathcal{A} -submodules of P_k . Furthermore, we have the following.

Proposition 2.12. *We have a direct summand decomposition of the \mathbb{F}_2 -vector spaces*

$$QP_k = QP_k^0 \oplus QP_k^+.$$

Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.

Notation 2.13. From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k ,

$$B_k^0(n) = B_k(n) \cap P_k^0, \quad B_k^+(n) = B_k(n) \cap P_k^+.$$

For a weight vector ω of degree n , we set

$$B_k(\omega) = B_k(n) \cap P_k(\omega), \quad B_k^+(\omega) = B_k^+(n) \cap P_k(\omega).$$

For a subset $S \subset P_k$, we denote $[S] = \{[f] : f \in S\}$. If $S \subset P_k(\omega)$, then we set

$$[S]_\omega = \{[f]_\omega : f \in S\}.$$

Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the bases of the \mathbb{F}_2 -vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

Now, we recall some notations and definitions in [21].

Denote

$$\mathcal{N}_k = \{(i; I); I = (i_1, i_2, \dots, i_r), 1 \leq i < i_1 < \dots < i_r \leq k, 0 \leq r < k\}.$$

Definition 2.14. Let $(i; I) \in \mathcal{N}_k$, let $r = \ell(I)$ be the length of I , and let u be an integer with $1 \leq u \leq r$. A monomial x in P_{k-1} is said to be u -compatible with $(i; I)$ if all of the following hold:

- i) $\nu_{i_1-1}(x) = \nu_{i_2-1}(x) = \dots = \nu_{i_{(u-1)}-1}(x) = 2^r - 1$,
- ii) $\nu_{i_u-1}(x) > 2^r - 1$,
- iii) $\alpha_{r-t}(\nu_{i_u-1}(x)) = 1, \forall t, 1 \leq t \leq u$,
- iv) $\alpha_{r-t}(\nu_{i_t-1}(x)) = 1, \forall t, u < t \leq r$.

Clearly, a monomial x can be u -compatible with a given $(i; I) \in \mathcal{N}_k$ for at most one value of u . By convention, x is 1-compatible with $(i; \emptyset)$.

For $1 \leq i \leq k$, define the homomorphism $f_i : P_{k-1} \rightarrow P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ x_{j+1}, & \text{if } i \leq j < k. \end{cases}$$

Definition 2.15. Let $(i; I) \in \mathcal{N}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1} + \dots + 2^{r-u}} \prod_{u < t \leq r} x_{i_t}^{2^{r-t}}$ for $r = \ell(I) > 0$, $x_{(\emptyset,1)} = 1$. For a monomial x in P_{k-1} , we define the monomial $\phi_{(i;I)}(x)$ in P_k by setting

$$\phi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1} f_i(x))/x_{(I,u)}, & \text{if there exists } u \text{ such that} \\ & x \text{ is } u\text{-compatible with } (i, I), \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have an \mathbb{F}_2 -linear map $\phi_{(i;I)} : P_{k-1} \rightarrow P_k$. In particular, $\phi_{(i;\emptyset)} = f_i$.

For a subset $B \subset P_{k-1}$, denote

$$\begin{aligned}\Phi^0(B) &= \bigcup_{1 \leq i \leq k} \phi_{(i; \emptyset)}(B) = \bigcup_{1 \leq i \leq k} f_i(B). \\ \Phi^+(B) &= \bigcup_{(i; I) \in \mathcal{N}_k, 0 < \ell(I) \leq k-1} \phi_{(i; I)}(B) \setminus P_k^0. \\ \Phi(B) &= \Phi^0(B) \bigcup \Phi^+(B).\end{aligned}$$

Clearly, we have

Proposition 2.16. *If B is a minimal set of generators for \mathcal{A} -module P_{k-1} in degree n , then $\Phi^0(B)$ is also a minimal set of generators for \mathcal{A} -module P_k^0 in degree n .*

Theorem 2.17. (See [21]) *Let $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$ with d_i positive integers such that $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1} \geq k-1 \geq 3$. If B is a minimal set of generators for \mathcal{A} -module P_{k-1} in degree n , then $\Phi(B)$ is also a minimal set of generators for \mathcal{A} -module P_k in degree n .*

Definition 2.18. For any $(i; I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i; I)} : P_k \rightarrow P_{k-1}$ of algebras by substituting

$$p_{(i; I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ \sum_{s \in I} x_{s-1}, & \text{if } j = i, \\ x_{j-1}, & \text{if } i < j \leq k. \end{cases}$$

Then, $p_{(i; I)}$ is a homomorphism of \mathcal{A} -modules. In particular, for $I = \emptyset$, $p_{(i; \emptyset)}(x_i) = 0$ and $p_{(i; I)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

Lemma 2.19. (See [12]) *If x is a monomial in P_k , then $p_{(i; I)}(x) \in P_{k-1}(\omega(x))$.*

Lemma 2.19 implies that if ω is a weight vector and $x \in P_k(\omega)$, then $p_{(i; I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i; I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$.

For a positive integer b , denote

$$\omega_{(k, b)} = ((k-1)^{(b)}), \quad \bar{\omega}_{(k, b)} = ((k-1)^{(b-1)}, k-3, 1).$$

Proposition 2.20. (See [12]) *Let d be a positive integer and let $p = \min\{k, d\}$. Then, the set*

$$B(d) := \{ [\phi_{(i; I)}(X^{2^d-1})]_{\omega_{(k, d)}} : (i; I) \in \mathcal{N}_{k, p} \}$$

is a basis of the \mathbb{F}_2 -vector space $QP_k(\omega_{(k, d)})$. Consequently

$$\dim QP_k(\omega_{(k, d)}) = \sum_{t=1}^p \binom{k}{t}.$$

3. PROOF OF MAIN THEOREM

According to a result in [21], $B_4(4(2^d-1))$ is the set consisting of 21 monomials, namely:

$$\begin{aligned}
v_{d,1} &= x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-1} & v_{d,2} &= x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^d-1} \\
v_{d,3} &= x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} & v_{d,4} &= x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} \\
v_{d,5} &= x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^d-1} & v_{d,6} &= x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3^{2^d-1} x_4^{2^{d-1}-1} \\
v_{d,7} &= x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} & v_{d,8} &= x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} \\
v_{d,9} &= x_1^{2^d-1} x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} & v_{d,10} &= x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^d-1} \\
v_{d,11} &= x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d-1}-1} & v_{d,12} &= x_1^{2^{d+1}-1} x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} \\
v_{d,13} &= x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^d-1} x_4^{2^d+2^{d-1}-1} & v_{d,14} &= x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^d+2^{d-1}-1} x_4^{2^d-1} \\
v_{d,15} &= x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^d+2^{d-1}-1} & v_{d,16} &= x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^{2^d+2^{d-1}-1} x_4^{2^d-1} \\
v_{d,17} &= x_1^{2^d-1} x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^d+2^{d-1}-1} & v_{d,18} &= x_1^{2^d-1} x_2^{2^d-1} x_3^{2^d+2^{d-1}-1} x_4^{2^{d-1}-1} \\
v_{d,19} &= x_1^{2^d-1} x_2^{2^d+2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^d-1} & v_{d,20} &= x_1^{2^d-1} x_2^{2^d+2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d-1}-1} \\
v_{d,21} &= x_1^{2^d-1} x_2^{2^d-1} x_3^{2^d-1} x_4^{2^d-1}.
\end{aligned}$$

Note that $B_4(4(2^d - 1)) = B_4(\omega_{(5,d)}) \cup B_4(\bar{\omega}_{(5,d)})$, where

$$B_4(\omega_{(5,d)}) = \{v_{d,21}\}, \quad B_4(\bar{\omega}_{(5,d)}) = \{v_{d,t} : 1 \leq t \leq 20\}.$$

Since $(P_5)_4 = (P_5^0)_4$, we obtain $B_5(4) = \Phi^0(B_4(4))$, $|B_5(4)| = 45$. It is easy to see that $|\Phi^0(B_4(\bar{\omega}_{(5,d)}))| = 100$, for $d > 1$. So, we obtain the following.

Proposition 3.1. *For any integer $d > 1$, $\dim QP_5^0(\bar{\omega}_{(5,d)}) = 100$.*

Lemma 3.2. *If x is an admissible monomial of degree $4(2^d - 1)$ in P_5 , then either $\omega(x) = \omega_{(5,d)}$ or $\omega(x) = \bar{\omega}_{(5,d)}$.*

Proof. We prove the lemma by induction on d . For $d = 1$, since $x \in B_5(4) = \Phi^0(B_4(4))$, we have either $\omega(x) = (4, 0) = \omega_{(5,1)}$ or $\omega(x) = (2, 1) = \bar{\omega}_{(5,1)}$. The lemma holds for $d = 1$.

Suppose $d > 1$ and the lemma holds for $1, 2, \dots, d-1$. Observe that the monomial $z = x_1^{2^{d+1}-1} x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1}$ is the minimal spike of degree $4(2^d - 1)$ in P_5 and $\omega(z) = \bar{\omega}_{(5,d)}$.

Since $4(2^d - 1)$ is even, one gets either $\omega_1(x) = 0$ or $\omega_1(x) = 2$ or $\omega_1(x) = 4$. If either $\omega_1(x) = 0$ or $\omega_1(x) = 2$ then $\omega(x) < \omega(z)$. By Theorem 2.9, x is hit. This contradicts the fact that x is admissible. So, $\omega_1(x) = 4$ and $x = X_i y^2$ with $1 \leq i \leq 5$ and y a monomial of degree $4(2^{d-1} - 1)$. Since x is admissible, according to Theorem 2.7, y is also admissible. Now, the lemma follows from the inductive hypothesis. \square

Since $n = 4(2^d - 1) = 2^{d+1} + 2^d + 2^{d-1} + 2^{d-1} - 4$, using Theorem 2.17, we get $\dim(QP_5)_{4(2^d-1)} = (2^5 - 1) \times 21 = 651$ for $d \geq 5$.

By Lemma 3.2, we have

$$(QP_5)_{4(2^d-1)} \cong QP_5(\omega_{(5,d)}) \oplus QP_5(\bar{\omega}_{(5,d)}).$$

Hence, combining Theorem 2.17 and Proposition 2.20, we obtain $\dim QP_5(\bar{\omega}_{(5,d)}) = 620$ for $d \geq 5$. So, we need only to compute $QP_5(\bar{\omega}_{(5,d)})$ for $2 \leq d \leq 4$.

For simplicity, we prove the theorem in detail for the case $d = 2$. The others can be proved by a similar computation. The admissible monomials $a_{d,t}$ of degree $4(2^d - 1)$ in P_5^+ are explicitly determined in Section 4.

3.1. The case $d = 2$.

For $d = 2$, we have $4(2^d - 1) = 12$. By a direct computation we see that

$$\Phi^+(B_4(\bar{\omega}_{(5,2)})) \cup \{x_1^3 x_2^4 x_3 x_4 x_5^3, x_1^3 x_2^4 x_3 x_4^3 x_5, x_1^3 x_2^4 x_3^3 x_4 x_5\}$$

is the set consisting of 75 monomials: $a_{2,t}$, $1 \leq t \leq 75$ (see Section 4).

Proposition 3.1.1. *The set $\{[a_{2,t}], 1 \leq t \leq 75\}$ is a basis of the \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,2)})$.*

The proof of the proposition is based on Theorem 2.7 and the following.

Lemma 3.1.2. *The following monomials are strictly inadmissible:*

- (i) $x_i^2 x_j x_\ell x_m x_n^3$, $i < j < \ell < m$,
- (ii) $x_i^2 x_j x_\ell^3 x_m^3 x_n^3$, $i < j$.

Here (i, j, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5)$.

Proof. We have

$$\begin{aligned} x_i^2 x_j x_\ell x_m x_n^3 &= x_i x_j^2 x_\ell x_m x_n^3 + x_i x_j x_\ell^2 x_m x_n^3 + x_i x_j x_\ell x_m^2 x_n^3 \\ &\quad + x_i x_j x_\ell x_m x_n^4 + Sq^1(x_i x_j x_\ell x_m x_n^3), \\ x_i^2 x_j x_\ell^3 x_m^3 x_n^3 &= x_i x_j^2 x_\ell^3 x_m^3 x_n^3 + x_i x_j x_\ell^4 x_m^3 x_n^3 + x_i x_j x_\ell^3 x_m^4 x_n^3 \\ &\quad + x_i x_j x_\ell^3 x_m^3 x_n^4 + Sq^1(x_i x_j x_\ell^3 x_m^3 x_n^3). \end{aligned}$$

The lemma follows from the above equalities. \square

Lemma 3.1.3. *The \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,2)})$ is spanned by the set*

$$\{[a_{2,t}], 1 \leq t \leq 75\}.$$

Proof. Let x be an admissible monomial in P_5 such that $\omega(x) = \bar{\omega}_{(5,2)}$. Then, $\omega_1(x) = 4$, $x = X_j y^2$ with $1 \leq j \leq 5$ and y a monomial of degree 4 in P_5 . Since x is admissible, according to Theorem 2.7, $y \in B_5(\bar{\omega}_{(5,1)})$.

Let $z \in B_5(\bar{\omega}_{(5,1)})$ and $1 \leq j \leq 5$. By a direct computation we see that if $X_j z^2 \neq a_{2,t}$, $\forall t, 1 \leq t \leq 75$, then there is a monomial w which is given in Lemma 3.1.2 such that $X_j z^2 = w z_1^{2^u}$ with suitable monomial $z_1 \in P_5$, and $u = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.7, $X_j z^2$ is inadmissible. Since $x = X_j y^2$ with $y \in B_5(\bar{\omega}_{(5,1)})$ and x is admissible, one gets $x = a_{2,t}$ for some t . The lemma is proved. \square

Lemma 3.1.4. *The set $\{[a_{2,t}], 1 \leq t \leq 75\}$ is linearly independent in $QP_5^+(\bar{\omega}_{(5,2)})$.*

Proof. Suppose there is a linear relation

$$\mathcal{S} = \sum_{t=1}^{75} \gamma_t a_{2,t} \equiv 0,$$

where $\gamma_t \in \mathbb{F}_2$ for all t , $1 \leq t \leq 75$.

Let J be a sequence of non-negative integers and $\gamma_j \in \mathbb{F}_2$ for $j \in J$. Denote by $\gamma_J = \sum_{j \in J} \gamma_j \in \mathbb{F}_2$. Based on Theorem 2.9, for $(i; I) \in \mathcal{N}_5$, we explicitly

compute $p_{(i;I)}(\mathcal{S})$ in terms of $v_{2,j}$, $1 \leq j \leq 20$. By computing from the relations $p_{(i;j)}(\mathcal{S}) \equiv 0$, $1 \leq i < j \leq 5$, we have

$$\begin{cases} \gamma_t = 0, \ t \in \mathbb{J}, \ \gamma_{55} = \gamma_{56} = \gamma_{57}, \\ \gamma_t = \gamma_4, \ t = 15, 26, 65, 68, 73, \\ \gamma_t = \gamma_5, \ t = 16, 34, 64, 69, 74, \\ \gamma_t = \gamma_6, \ t = 27, 36, 66, 67, 75, \\ \gamma_t = \gamma_{17}, \ t = 29, 38, 70, 71, 72, \\ \gamma_{\{6,17,43,55\}} = \gamma_{\{5,17,47,55\}} = 0, \\ \gamma_{\{4,17,51,55\}} = \gamma_{\{5,6,43,47\}} = 0, \\ \gamma_{\{4,6,43,51\}} = \gamma_{\{4,5,47,51\}} = 0. \end{cases} \quad (3.1)$$

Here $\mathbb{J} = \{1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19, 20, 21, 22, 23, 24, 25, 28, 30, 31, 32, 33, 35, 37, 39, 40, 41, 42, 44, 45, 46, 48, 49, 50, 52, 53, 54, 58, 59, 60, 61, 62, 63\}$. Then, computing from the relations $p_{(1;(2,3))}(\mathcal{S}) \equiv 0$ and $p_{(1;(2,4))}(\mathcal{S}) \equiv 0$ gives

$$\begin{cases} \gamma_t = 0, \ t \in \{4, 5, 6, 43, 47, 55\}, \\ \gamma_{\{4,5,47,51\}} = 0. \end{cases} \quad (3.2)$$

By combining the relations (3.1) and (3.2), we obtain $\gamma_t = 0$, $1 \leq t \leq 75$. The lemma is proved. \square

From Propositions 2.20, 3.1 and 3.1.1, we have $\dim(QP_5)_{12} = 190$.

3.2. The case $d = 3$.

For $d = 3$, we have $4(2^d - 1) = 28$. Denote by C the set of the following monomials:

$$\begin{array}{lll} x_1^7 x_2^9 x_3^2 x_4^3 x_5^7 & x_1^7 x_2^9 x_3^2 x_4^7 x_5^3 & x_1^7 x_2^9 x_3^3 x_4^2 x_5^7 \\ x_1^7 x_2^9 x_3^3 x_4^4 x_5^2 & x_1^7 x_2^9 x_3^3 x_4^2 x_5^3 & x_1^7 x_2^9 x_3^3 x_4^3 x_5^2 \\ x_1^3 x_2^7 x_3^{11} x_4^4 x_5^3 & x_1^7 x_2^3 x_3^{11} x_4^4 x_5^3 & x_1^7 x_2^{11} x_3^3 x_4^4 x_5^3 \\ x_1^7 x_2^9 x_3^3 x_4^3 x_5^6 & x_1^7 x_2^9 x_3^3 x_4^6 x_5^3 & x_1^3 x_2^7 x_3^7 x_4^8 x_5^3 \\ x_1^3 x_2^7 x_3^8 x_4^3 x_5^7 & x_1^3 x_2^7 x_3^8 x_4^7 x_5^3 & x_1^7 x_2^3 x_3^7 x_4^8 x_5^3 \\ x_1^7 x_2^3 x_3^8 x_4^3 x_5^7 & x_1^7 x_2^3 x_3^8 x_4^7 x_5^3 & x_1^7 x_2^7 x_3^3 x_4^8 x_5^3 \\ x_1^7 x_2^7 x_3^8 x_4^3 x_5^3 & & \end{array}$$

A direct computation shows that $\Phi^+(B_4(\bar{\omega}_{(5,3)})) \cup C$ is the set of 355 monomials: $a_{3,t}$, $1 \leq t \leq 355$ (see Section 4).

Proposition 3.2.1. *Under the above notations, the set $[\Phi^+(B_4(\bar{\omega}_{(5,3)})) \cup C]$ is a basis of the \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,3)})$.*

We prove the proposition by proving some lemmas.

Lemma 3.2.2. *If (i, j, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5)$ such that $i < j$, then the monomial $x_i^3 x_j^4 x_\ell^7 x_m^7 x_n^7$ is strictly inadmissible.*

Proof. By a direct computation, we have

$$\begin{aligned}
x_i^3 x_j^4 x_\ell^7 x_m^7 x_n^7 &= x_i^2 x_j^5 x_\ell^7 x_m^7 x_n^7 + x_i^2 x_j^3 x_\ell^9 x_m^7 x_n^7 \\
&\quad + x_i^2 x_j^3 x_\ell^7 x_m^9 x_n^7 + x_i^2 x_j^3 x_\ell^7 x_m^7 x_n^9 \\
&\quad + x_i^3 x_j^3 x_\ell^8 x_m^7 x_n^7 + x_i^3 x_j^3 x_\ell^7 x_m^8 x_n^7 \\
&\quad + x_i^3 x_j^3 x_\ell^7 x_m^7 x_n^8 + x_i^2 x_j^3 x_\ell^8 x_m^8 x_n^7 \\
&\quad + x_i^2 x_j^3 x_\ell^8 x_m^7 x_n^8 + x_i^2 x_j^4 x_\ell^7 x_m^8 x_n^8 \\
&\quad + Sq^1(x_i^3 x_j^3 x_\ell^7 x_m^7 x_n^7 + x_i^2 x_j^4 x_\ell^7 x_m^7 x_n^7) \\
&\quad + Sq^2(x_i^2 x_j^3 x_\ell^7 x_m^7 x_n^7).
\end{aligned}$$

The lemma is proved. \square

By a similar computation, one gets the following.

Lemma 3.2.3. *The following monomials are strictly inadmissible:*

- (i) $x_i^3 x_j^4 x_\ell^3 x_m^7 x_n^7$, $i < j < 4$, $2 < \ell < m$,
- (ii) $x_1^7 x_2^9 x_3^6 x_4^3 x_5^3$, $x_1^7 x_2^8 x_3^3 x_4^7 x_5^7$, $x_1^7 x_2^8 x_3^3 x_4^7 x_5^3$, $x_1^7 x_2^8 x_3^7 x_4^3 x_5^3$.

Here (i, j, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5)$.

Lemma 3.2.4. *The \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,3)})$ is spanned by the set*

$$[\Phi^+(B_4(\bar{\omega}_{(5,3)})) \cup C].$$

Proof. Let x be an admissible monomial in P_5 such that $\omega(x) = \bar{\omega}_{(5,3)}$. Then, $\omega_1(x) = 4$ and $x = X_j y^2$ with $1 \leq j \leq 5$ and y a monomial of degree 12 in P_5 . Since x is admissible, according to Theorem 2.7, $y \in B_5(\bar{\omega}_{(5,2)})$.

A direct computation shows that if $z \in B_5(\bar{\omega}_{(5,2)})$, $1 \leq j \leq 5$ and $X_j z^2 \neq a_{3,t}$, $\forall t, 1 \leq t \leq 355$, then there exists a monomial w which is given in one of Lemmas 3.1.2(ii), 3.2.2, 3.2.3 such that $X_j z^2 = w z_1^{2^r}$ with a monomial $z_1 \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.7, $X_j z^2$ is inadmissible. Since $x = X_j y^2$ with $y \in B_5(\bar{\omega}_{(5,2)})$ and x is admissible, one can see that $x = a_{3,t}$ for suitable t . The lemma follows. \square

Lemma 3.2.5. *The set $[\Phi^+(B_4(\bar{\omega}_{(5,3)})) \cup C]$ is linearly independent in $QP_5^+(\bar{\omega}_{(5,3)})$.*

Proof. Suppose there is a linear relation

$$\mathcal{S} = \sum_{t=1}^{355} \gamma_t a_{3,t} \equiv 0,$$

where $\gamma_t \in \mathbb{F}_2$ for all $1 \leq t \leq 355$. Using Theorem 2.9, we explicitly compute $p_{(i;I)}(S)$, $(i;I) \in \mathcal{N}_5$, in terms of $v_{3,j}$, $1 \leq j \leq 20$. By a direct computation from the relations $p_{(i;I)}(S) \equiv 0$, $(i;I) \in \mathcal{N}_5$ with $0 < \ell(I) \leq 3$, we obtain $\gamma_t = 0$ for $1 \leq t \leq 355$. \square

By using Propositions 2.20, 3.1 and 3.2.1, we get $\dim(QP_5)_{28} = 480$.

3.3. The case $d = 4$.

For $d = 4$, we have $4(2^d - 1) = 60$. Denote by D the set of the following monomials:

$$\begin{array}{lll}
x_1^7 x_2^7 x_3^{15} x_4^{15} x_5^{16} & x_1^7 x_2^7 x_3^{15} x_4^{23} x_5^8 & x_1^7 x_2^{15} x_3^7 x_4^{15} x_5^{16} \\
x_1^7 x_2^{15} x_3^7 x_4^{23} x_5^8 & x_1^7 x_2^{15} x_3^{15} x_4^7 x_5^{16} & x_1^7 x_2^{15} x_3^{15} x_4^{16} x_5^7 \\
x_1^7 x_2^{15} x_3^{15} x_4^{17} x_5^6 & x_1^7 x_2^{15} x_3^{17} x_4^7 x_5^{14} & x_1^7 x_2^{15} x_3^{23} x_4^7 x_5^8 \\
x_1^7 x_2^{15} x_3^{23} x_4^9 x_5^6 & x_1^{15} x_2^7 x_3^7 x_4^{15} x_5^{16} & x_1^{15} x_2^7 x_3^7 x_4^{23} x_5^8 \\
x_1^{15} x_2^7 x_3^{15} x_4^7 x_5^{16} & x_1^{15} x_2^7 x_3^{15} x_4^{16} x_5^7 & x_1^{15} x_2^7 x_3^{15} x_4^{17} x_5^6 \\
x_1^{15} x_2^7 x_3^{17} x_4^7 x_5^{14} & x_1^{15} x_2^7 x_3^{23} x_4^7 x_5^8 & x_1^{15} x_2^7 x_3^{23} x_4^9 x_5^6 \\
x_1^{15} x_2^{15} x_3^7 x_4^7 x_5^{16} & x_1^{15} x_2^{15} x_3^7 x_4^{16} x_5^7 & x_1^{15} x_2^{15} x_3^7 x_4^{17} x_5^6 \\
x_1^{15} x_2^{15} x_3^{17} x_4^7 x_5^7 & x_1^{15} x_2^{15} x_3^{17} x_4^7 x_5^6 & x_1^{15} x_2^{19} x_3^7 x_4^7 x_5^{14} \\
x_1^{15} x_2^{19} x_3^7 x_4^5 x_5^{14} & x_1^{15} x_2^{19} x_3^7 x_4^{13} x_5^6 & x_1^{15} x_2^{23} x_3^7 x_4^7 x_5^8 \\
x_1^{15} x_2^{23} x_3^7 x_4^9 x_5^6 & &
\end{array}$$

A direct computation shows that $\Phi^+(B_4(\bar{\omega}_{(5,4)})) \cup D$ is the set of 520 monomials: $a_{4,t}$, $1 \leq t \leq 520$ (see Section 4).

Proposition 3.3.1. *Under the above notation, the set $[\Phi^+(B_4(\bar{\omega}_{(5,4)})) \cup D]$ is a basis of the \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,4)})$.*

We prepare some lemmas for the proof of this proposition. The following lemma is proved by a direct computation.

Lemma 3.3.2. *If (i, j, ℓ, m, n) is a permutation of $(1, 2, 3, 4, 5)$ such that $i < j < \ell < m$, then the monomial $x_i^7 x_j^8 x_\ell^7 x_m^{15} x_n^{15}$ is strictly inadmissible.*

Lemma 3.3.3. *The following monomials are strictly inadmissible:*

$$\begin{array}{lll}
x_1^7 x_2^7 x_3^{15} x_4^{16} x_5^{15} & x_1^7 x_2^{15} x_3^7 x_4^{16} x_5^{15} & x_1^7 x_2^{15} x_3^{16} x_4^7 x_5^{15} \\
x_1^7 x_2^{15} x_3^{16} x_4^{15} x_5^7 & x_1^7 x_2^{15} x_3^{17} x_4^6 x_5^{15} & x_1^7 x_2^{15} x_3^{17} x_4^{14} x_5^7 \\
x_1^7 x_2^{15} x_3^{17} x_4^{15} x_5^6 & x_1^{15} x_2^7 x_3^7 x_4^{16} x_5^{15} & x_1^{15} x_2^7 x_3^{16} x_4^7 x_5^{15} \\
x_1^{15} x_2^7 x_3^{16} x_4^{15} x_5^7 & x_1^{15} x_2^7 x_3^{17} x_4^6 x_5^{15} & x_1^{15} x_2^7 x_3^{17} x_4^{14} x_5^7 \\
x_1^{15} x_2^7 x_3^{17} x_4^{15} x_5^6 & x_1^{15} x_2^{15} x_3^{16} x_4^7 x_5^7 & x_1^{15} x_2^{19} x_3^5 x_4^6 x_5^{15} \\
x_1^{15} x_2^{19} x_3^5 x_4^{14} x_5^7 & x_1^{15} x_2^{19} x_3^5 x_4^{15} x_5^6 & x_1^{15} x_2^{19} x_3^{15} x_4^5 x_5^6.
\end{array}$$

Proof. We prove the lemma for $x = x_1^7 x_2^7 x_3^{15} x_4^{16} x_5^{15}$. The others can be prove by a similar computation.

We have $\omega(x) = \bar{\omega}_{(5,4)}$. By a direct computation using the Cartan formula, we have

$$\begin{aligned}
x &= x_1^7 x_2^7 x_3^{15} x_4^{15} x_5^{16} + x_1^7 x_2^7 x_3^8 x_4^{23} x_5^{15} + x_1^7 x_2^7 x_3^8 x_4^{15} x_5^{23} \\
&\quad + x_1^4 x_2^7 x_3^{19} x_4^{15} x_5^{15} + x_1^4 x_2^7 x_3^{15} x_4^{19} x_5^{15} + x_1^4 x_2^7 x_3^{15} x_4^{15} x_5^{19} \\
&\quad + x_1^7 x_2^6 x_3^{17} x_4^{15} x_5^{15} + x_1^7 x_2^6 x_3^{15} x_4^{17} x_5^{15} + x_1^7 x_2^6 x_3^{15} x_4^{15} x_5^{17} \\
&\quad + x_1^5 x_2^6 x_3^{19} x_4^{15} x_5^{15} + x_1^5 x_2^6 x_3^{15} x_4^{19} x_5^{15} + x_1^5 x_2^6 x_3^{15} x_4^{15} x_5^{19} \\
&\quad + Sq^1(x_1^7 x_2^7 x_3^{15} x_4^{15} x_5^{15} + x_1^3 x_2^{11} x_3^{15} x_4^{15} x_5^{15}) \\
&\quad + Sq^2(x_1^7 x_2^6 x_3^{15} x_4^{15} x_5^{15} + x_1^3 x_2^{10} x_3^{15} x_4^{15} x_5^{15}) \\
&\quad + Sq^4(x_1^4 x_2^7 x_3^{15} x_4^{15} x_5^{15} + x_1^5 x_2^6 x_3^{15} x_4^{15} x_5^{15}) \\
&\quad + Sq^8(x_1^7 x_2^7 x_3^8 x_4^{15} x_5^{15}) \pmod{(P_5^-(\bar{\omega}_{(5,4)})}.
\end{aligned}$$

Hence, x is strictly inadmissible. \square

Lemma 3.3.4. *The \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,4)})$ is spanned by the set*

$$[\Phi^+(B_4(\bar{\omega}_{(5,4)})) \cup D].$$

Proof. Let x be an admissible monomial in P_5 such that $\omega(x) = \bar{\omega}_{(5,4)}$. Then, $\omega_1(x) = 4$, and $x = X_j y^2$ with $1 \leq j \leq 5$ and y a monomial in P_5 such that $\omega(y) = \bar{\omega}_{(5,3)}$. Since x is admissible, according to Theorem 2.7, we have $y \in B_5(\bar{\omega}_{(5,3)})$.

By a direct computation we can verify that for any $z \in B_5(\bar{\omega}_{(5,3)})$, $1 \leq j \leq 5$, such that $X_j z^2 \neq a_{4,t}$, $\forall t$, $1 \leq t \leq 520$, there is a monomial w which is given in one of Lemmas 3.1.2(ii), 3.2.2, 3.3.2, 3.3.3 such that $X_j z^2 = wu^{2^r}$ with a monomial $u \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.7, $X_j z^2$ is inadmissible. Since $x = X_j y^2$ is admissible and $y \in B_5(\bar{\omega}_{(5,3)})$, one gets $x = a_{4,t}$ for some t . This proves the lemma. \square

Lemma 3.3.5. *The set $[\Phi^+(B_4(\bar{\omega}_{(5,4)})) \cup D]$ is linearly independent in the \mathbb{F}_2 -vector space $QP_5^+(\bar{\omega}_{(5,4)})$.*

Proof. Suppose there is a linear relation

$$\mathcal{S} = \sum_{t=1}^{520} \gamma_t a_{4,t} \equiv 0,$$

where $\gamma_t \in \mathbb{F}_2$ for all t , $1 \leq t \leq 520$. By a direct computation using Theorem 2.9, we express $p_{(i;I)}(\mathcal{S})$ in terms of $v_{4,j}$, $1 \leq j \leq 20$. Computing directly from the relations

$$p_{(i;I)}(S) \equiv 0, \quad \forall (i;I) \in \mathcal{N}_5, \quad \ell(I) > 0,$$

we get $\gamma_t = 0$ for all t . The lemma is proved. \square

By using Propositions 2.20, 3.1 and 3.3.1, we get $\dim(QP_5)_{60} = 650$. The Main Theorem is completely proved.

4. THE ADMISSIBLE MONOMIALS OF DEGREE $4(2^d - 1)$ IN P_5

In this section, we list all the admissible monomials of degree $4(2^d - 1)$ in P_5 . Recall that

$$B_5(4(2^d - 1)) = \Phi^0(B_4(4(2^d - 1))) \cup B_5^+(\omega_{(5,d)}) \cup B_5^+(\bar{\omega}_{(5,d)}),$$

where $B_4(4(2^d - 1)) = \{v_{d,j} : 1 \leq j \leq 21\}$ and

$$B_5^+(\omega_{(5,d)}) = \{\phi_{(i;I)}(v_{d,21}) : (i;I) \in \mathcal{N}_5, 1 \leq \ell(I) < \min\{5, d\}\}.$$

Set $b_d = |B_5^+(\bar{\omega}_{(5,d)})|$ and $B_5^+(\bar{\omega}_{(5,d)}) = \{a_{d,t} : 1 \leq t \leq b_d\}$. We have $b_1 = 0, b_2 = 75, b_3 = 355$ and $b_d = 520$ for $d \geq 4$. The admissible monomials $a_{d,t}$, $1 \leq t \leq b_d$, are determined as follows:

For $d \geq 2$,

- | | |
|---|---|
| 1. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d+1}-1}$ | 2. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-2}$ |
| 3. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2} x_5^{2^d-1}$ | 4. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^d-2}$ |
| 5. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-1}$ | 6. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-2} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$ |
| 7. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$ | 8. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-2} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$ |
| 9. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$ | 10. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$ |
| 11. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$ | 12. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$ |

13. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
14. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
15. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
16. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
17. $x_1 x_2^{2^d-2} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-1}$
18. $x_1 x_2^{2^d-2} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
19. $x_1 x_2^{2^d-2} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$
20. $x_1 x_2^{2^d-2} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$
21. $x_1 x_2^{2^d-2} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-1}$
22. $x_1 x_2^{2^d-2} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-1}$
23. $x_1 x_2^{2^d-2} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-1}$
24. $x_1 x_2^{2^d-2} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-1}$
25. $x_1 x_2^{2^d-2} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$
26. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
27. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
28. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
29. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
30. $x_1 x_2^{2^d-1} x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-1}$
31. $x_1 x_2^{2^d-1} x_3^{2^d-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-1}$
32. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
33. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
34. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
35. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
36. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$
37. $x_1 x_2^{2^{d+1}-2} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-1}$
38. $x_1 x_2^{2^{d+1}-2} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-1}$
39. $x_1 x_2^{2^{d+1}-2} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$
40. $x_1 x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
41. $x_1 x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
42. $x_1 x_2^{2^{d+1}-1} x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$
43. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$
44. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
45. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
46. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
47. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
48. $x_1^{2^d-1} x_2 x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-1}$
49. $x_1^{2^d-1} x_2 x_3^{2^d-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-1}$
50. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
51. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
52. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-1}$
53. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3 x_4^{2^{d-1}-1} x_5^{2^d-2}$
54. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3 x_4^{2^d-2} x_5^{2^{d-1}-1}$
55. $x_1^{2^{d+1}-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
56. $x_1^{2^{d+1}-1} x_2 x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
57. $x_1^{2^{d+1}-1} x_2 x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$

For $d = 2$,

58. $x_1 x_2 x_3^3 x_4^3 x_5^4$
59. $x_1 x_2 x_3^3 x_4^4 x_5^3$
60. $x_1 x_2^3 x_3 x_4^3 x_5^4$
61. $x_1 x_2^3 x_3 x_4^4 x_5^3$
62. $x_1 x_2^3 x_3^3 x_4^4 x_5$
63. $x_1 x_2^3 x_3^3 x_4^4 x_5$
64. $x_1^3 x_2 x_3 x_4^3 x_5^4$
65. $x_1^3 x_2 x_3 x_4^4 x_5^3$
66. $x_1^3 x_2 x_3^3 x_4^4 x_5$
67. $x_1^3 x_2 x_3^3 x_4^4 x_5$
68. $x_1^3 x_2^3 x_3 x_4^3 x_5^4$
69. $x_1^3 x_2^3 x_3 x_4^4 x_5^3$
70. $x_1^3 x_2^3 x_3 x_4^4 x_5$
71. $x_1^3 x_2^3 x_3 x_4^4 x_5$
72. $x_1^3 x_2^3 x_3^3 x_4 x_5$
73. $x_1^3 x_2^3 x_3^3 x_4 x_5$
74. $x_1^3 x_2^3 x_3^3 x_4 x_5$
75. $x_1^3 x_2^3 x_3^3 x_4 x_5$

For $d \geq 3$,

58. $x_1 x_2^{2^{d-1}-2} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-1}$
59. $x_1 x_2^{2^{d-1}-2} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^d-1}$
60. $x_1 x_2^{2^{d-1}-2} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-1}$
61. $x_1 x_2^{2^{d-1}-2} x_3^{2^d-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
62. $x_1 x_2^{2^{d-1}-2} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$
63. $x_1 x_2^{2^{d-1}-2} x_3^{2^d-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$
64. $x_1 x_2^{2^{d-1}-2} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^d-1}$
65. $x_1 x_2^{2^{d-1}-2} x_3^{2^{d+1}-1} x_4^{2^d-1} x_5^{2^{d-1}-1}$
66. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d+1}-1}$
67. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-2} x_4^{2^{d+1}-1} x_5^{2^d-1}$
68. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
69. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
70. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
71. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
72. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
73. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
74. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-1}$
75. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
76. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-2} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$
77. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-2} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$

78. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
80. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
82. $x_1 x_2^{2^d-1} x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$
84. $x_1 x_2^{2^d-1} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$
86. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$
88. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
90. $x_1 x_2^{2^{d+1}-1} x_3^{2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
92. $x_1 x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$
94. $x_1 x_2^{2^{d+1}-1} x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
96. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d+1}-2}$
98. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
100. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
102. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
104. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
106. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
108. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
110. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
112. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
114. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-1}$
116. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$
118. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
120. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
122. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
124. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
126. $x_1 x_2^{2^d-3} x_3^{2^d-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-1}$
128. $x_1 x_2^{2^d-3} x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$
130. $x_1 x_2^{2^d-3} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$
132. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
134. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$
136. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
138. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2} x_4^{2^{d-1}-1} x_5^{2^{d-1}-1}$
140. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
142. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
144. $x_1 x_2^{2^d-1} x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$
146. $x_1 x_2^{2^d-1} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$
148. $x_1 x_2^{2^d-1} x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2^{d-1}-2}$
150. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$
152. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$
154. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
156. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-3} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
79. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
81. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
83. $x_1 x_2^{2^d-1} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-2}$
85. $x_1 x_2^{2^d-1} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
87. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
89. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
91. $x_1 x_2^{2^{d+1}-1} x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
93. $x_1 x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
95. $x_1 x_2^{2^{d+1}-1} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
97. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-3} x_5^{2^d-2}$
99. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
101. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
103. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
105. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-3} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
107. $x_1 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
109. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$
111. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$
113. $x_1 x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
115. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
117. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-2} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$
119. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
121. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
123. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
125. $x_1 x_2^{2^d-3} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
127. $x_1 x_2^{2^d-3} x_3^{2^d-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-1}$
129. $x_1 x_2^{2^d-3} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-2}$
131. $x_1 x_2^{2^d-3} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
133. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
135. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
137. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
139. $x_1 x_2^{2^d-3} x_3^{2^{d+1}-1} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
141. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
143. $x_1 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
145. $x_1 x_2^{2^d-1} x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-2}$
147. $x_1 x_2^{2^d-1} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
149. $x_1 x_2^{2^d-1} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
151. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
153. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
155. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
157. $x_1 x_2^{2^d-1} x_3^{2^{d+1}-3} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$

158. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
 160. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$
 162. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$
 164. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
 166. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
 168. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
 170. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
 172. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d-1}-2} x_4^{2^{d+1}-1} x_5^{2^d-1}$
 174. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-2}$
 176. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^d-2}$
 178. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-2} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
 180. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-2} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$
 182. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
 184. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
 186. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
 188. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
 190. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d+1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
 192. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
 194. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d+1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 196. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3 x_4^{2^d-1} x_5^{2^{d+1}-2}$
 198. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3 x_4^{2^{d+1}-1} x_5^{2^d-2}$
 200. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4 x_5^{2^d-2}$
 202. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
 204. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
 206. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
 208. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
 210. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^d-1} x_4 x_5^{2^{d+1}-2^{d-1}-2}$
 212. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^{d+1}-1} x_4 x_5^{2^{d-1}-2}$
 214. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3 x_4^{2^{d-1}-1} x_5^{2^d-2}$
 216. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3 x_4^{2^d-1} x_5^{2^{d-1}-2}$
 218. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3^{2^d-1} x_4 x_5^{2^{d-1}-2}$
 220. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
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 228. $x_1^{2^d-1} x_2 x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-2}$
 230. $x_1^{2^d-1} x_2 x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
 232. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
 234. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 236. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
 159. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
 161. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^{d-1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
 163. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 165. $x_1^3 x_2^{2^{d+1}-3} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
 167. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
 169. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
 171. $x_1^{2^{d-1}-1} x_2 x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d+1}-1}$
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 179. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$
 181. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
 183. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
 185. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
 187. $x_1^{2^{d-1}-1} x_2 x_3^{2^d-1} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
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 195. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3 x_4^{2^d-2} x_5^{2^{d+1}-1}$
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 199. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3^{2^d-1} x_4 x_5^{2^{d+1}-2}$
 201. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
 203. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
 205. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
 207. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
 209. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^{d-1}-1} x_4 x_5^{2^{d+1}-2}$
 211. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4 x_5^{2^d-2}$
 213. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3 x_4^{2^{d-1}-2} x_5^{2^d-1}$
 215. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3 x_4^{2^d-2} x_5^{2^{d-1}-1}$
 217. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4 x_5^{2^d-2}$
 219. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^{d+1}-1}$
 221. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$
 223. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
 225. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
 227. $x_1^{2^d-1} x_2 x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$
 229. $x_1^{2^d-1} x_2 x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$
 231. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
 233. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$
 235. $x_1^{2^d-1} x_2 x_3^{2^{d+1}-1} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
 237. $x_1^{2^d-1} x_2 x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$

238. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
 240. $x_1^{2^d-1} x_2^3 x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$
 242. $x_1^{2^d-1} x_2^3 x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$
 244. $x_1^{2^d-1} x_2^3 x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
 246. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
 248. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 250. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-3} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
 252. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3 x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
 254. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3 x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
 256. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3 x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
 258. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3 x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
 260. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^{2^{d-1}-1} x_4 x_5^{2^{d+1}-2}$
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 264. $x_1^{2^d-1} x_2^{2^d-1} x_3 x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$
 266. $x_1^{2^d-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$
 268. $x_1^{2^d-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
 270. $x_1^{2^d-1} x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-1} x_4 x_5^{2^{d-1}-2}$
 272. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3 x_4^{2^d-1} x_5^{2^{d-1}-2}$
 274. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3^{2^{d-1}-1} x_4 x_5^{2^d-2}$
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 239. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-1} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
 241. $x_1^{2^d-1} x_2^3 x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-2}$
 243. $x_1^{2^d-1} x_2^3 x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
 245. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$
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 249. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
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 255. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3 x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
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 267. $x_1^{2^d-1} x_2^{2^d-1} x_3 x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
 269. $x_1^{2^d-1} x_2^{2^d-1} x_3^{2^{d-1}-1} x_4 x_5^{2^{d+1}-2^{d-1}-2}$
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 273. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3 x_4^{2^d-3} x_5^{2^{d-1}-2}$
 275. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3^{2^d-1} x_4 x_5^{2^{d-1}-2}$
 277. $x_1^{2^d-1} x_2^{2^{d+1}-1} x_3 x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
 279. $x_1^{2^{d+1}-1} x_2 x_3^{2^{d-1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$
 281. $x_1^{2^{d+1}-1} x_2 x_3^{2^{d-1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$
 283. $x_1^{2^{d+1}-1} x_2 x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
 285. $x_1^{2^{d+1}-1} x_2 x_3^{2^d-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
 287. $x_1^{2^{d+1}-1} x_2 x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
 289. $x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3 x_4^{2^{d-1}-1} x_5^{2^d-2}$
 291. $x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3 x_4^{2^d-1} x_5^{2^{d-1}-2}$
 293. $x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3^{2^d-1} x_4 x_5^{2^{d-1}-2}$
 295. $x_1^{2^{d+1}-1} x_2^{2^d-1} x_3 x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$

For $d = 3$,

297. $x_1^3 x_2^3 x_3^3 x_4^4 x_5^{15}$ 298. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^{12}$ 299. $x_1^3 x_2^3 x_3^3 x_4^{12} x_5^7$ 300. $x_1^3 x_2^3 x_3^3 x_4^{15} x_5^4$
 301. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{15}$ 302. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^{11}$ 303. $x_1^3 x_2^3 x_3^3 x_4^{11} x_5^7$ 304. $x_1^3 x_2^3 x_3^3 x_4^{15} x_5^3$
 305. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{12}$ 306. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^{11}$ 307. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^8$ 308. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^7$
 309. $x_1^3 x_2^3 x_3^3 x_4^{11} x_5^4$ 310. $x_1^3 x_2^3 x_3^3 x_4^{12} x_5^3$ 311. $x_1^3 x_2^3 x_3^3 x_4^{12} x_5^7$ 312. $x_1^3 x_2^3 x_3^3 x_4^{12} x_5^3$
 313. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{15}$ 314. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{15}$ 315. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{12}$ 316. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{11}$
 317. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^8$ 318. $x_1^3 x_2^3 x_3^3 x_4^7 x_5^7$ 319. $x_1^3 x_2^3 x_3^3 x_4^{11} x_5^4$ 320. $x_1^3 x_2^3 x_3^3 x_4^{12} x_5^3$
 321. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^8$ 322. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^7$ 323. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{12}$ 324. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^{11}$
 325. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^4$ 326. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$ 327. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^4$ 328. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$
 329. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^8$ 330. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^7$ 331. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^4$ 332. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$
 333. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^8$ 334. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^7$ 335. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^4$ 336. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$
 337. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^4$ 338. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$ 339. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^8$ 340. $x_1^3 x_2^3 x_3^3 x_4^3 x_5^3$

341. $x_1^7 x_2^7 x_3^8 x_4^3 x_5^3$ 342. $x_1^7 x_2^7 x_3^9 x_4^2 x_5^3$ 343. $x_1^7 x_2^7 x_3^9 x_4^3 x_5^2$ 344. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^7$
 345. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^3$ 346. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^6$ 347. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^3$ 348. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^3$
 349. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^2$ 350. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^3$ 351. $x_1^7 x_2^9 x_3^9 x_4^3 x_5^2$ 352. $x_1^7 x_2^{11} x_3^3 x_4^3 x_5^4$
 353. $x_1^7 x_2^{11} x_3^3 x_4^3 x_5^3$ 354. $x_1^{15} x_2^3 x_3^3 x_4^3 x_5^4$ 355. $x_1^{15} x_2^3 x_3^3 x_4^3 x_5^3$

For $d \geq 4$,

297. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d-1}-2} x_4^{2^d-1} x_5^{2^{d+1}-1}$ 298. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d-1}-2} x_4^{2^{d+1}-1} x_5^{2^d-1}$
 299. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d-1}-1} x_4^{2^d-2} x_5^{2^{d+1}-1}$ 300. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d-1}-1} x_4^{2^d-1} x_5^{2^{d+1}-2}$
 301. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2} x_5^{2^d-1}$ 302. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d-1}-1} x_4^{2^{d+1}-1} x_5^{2^d-2}$
 303. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-2} x_4^{2^{d+1}-1} x_5^{2^{d+1}-1}$ 304. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-2} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-1}$
 305. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-2} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-1}$ 306. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-2} x_4^{2^{d+1}-1} x_5^{2^{d-1}-1}$
 307. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$ 308. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
 309. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$ 310. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
 311. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$ 312. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
 313. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$ 314. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^d-1} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
 315. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d+1}-2} x_4^{2^{d-1}-1} x_5^{2^d-1}$ 316. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d+1}-2} x_4^{2^d-1} x_5^{2^{d-1}-1}$
 317. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d+1}-1} x_4^{2^{d-1}-2} x_5^{2^d-1}$ 318. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d+1}-1} x_4^{2^{d-1}-1} x_5^{2^d-2}$
 319. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d+1}-1} x_4^{2^d-2} x_5^{2^{d-1}-1}$ 320. $x_1^3 x_2^{2^{d-1}-3} x_3^{2^{d+1}-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 321. $x_1^3 x_2^{2^{d-1}-1} x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d+1}-1}$ 322. $x_1^3 x_2^{2^{d-1}-1} x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d+1}-2}$
 323. $x_1^3 x_2^{2^{d-1}-1} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2} x_5^{2^d-1}$ 324. $x_1^3 x_2^{2^{d-1}-1} x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^d-2}$
 325. $x_1^3 x_2^{2^{d-1}-1} x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 326. $x_1^3 x_2^{2^{d-1}-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
 327. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$ 328. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
 329. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$ 330. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
 331. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$ 332. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
 333. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$ 334. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
 335. $x_1^3 x_2^{2^d-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 336. $x_1^3 x_2^{2^d-1} x_3^{2^{d+1}-1} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
 337. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$ 338. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
 339. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$ 340. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 341. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$ 342. $x_1^3 x_2^{2^{d+1}-1} x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
 343. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 344. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^d-5} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
 345. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^d-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$ 346. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^d-5} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
 347. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^{d+1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$ 348. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^{d+1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$
 349. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$ 350. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^{d+1}-2}$
 351. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$ 352. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
 353. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$ 354. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
 355. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$ 356. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
 357. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 358. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
 359. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$ 360. $x_1^7 x_2^{2^d-5} x_3^{2^{d-1}-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
 361. $x_1^7 x_2^{2^d-5} x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-2^{d-1}-1}$ 362. $x_1^7 x_2^{2^d-5} x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2^{d-1}-2}$
 363. $x_1^7 x_2^{2^d-5} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^{d-1}-1}$ 364. $x_1^7 x_2^{2^d-5} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^{d-1}-2}$
 365. $x_1^7 x_2^{2^d-5} x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2^{d-1}-2}$ 366. $x_1^7 x_2^{2^d-5} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
 367. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$ 368. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$

369. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$
371. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
373. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-3} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
375. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-1} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
377. $x_1^7 x_2^{2^d-1} x_3^{2^d-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
379. $x_1^7 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$
381. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$
383. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$
385. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
387. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
389. $x_1^{15} x_2^{2^{d+1}-9} x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
391. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d+1}-2}$
393. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^d-2}$
395. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d-1}-1} x_4^{2^{d+1}-3} x_5^{2^d-2}$
397. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
399. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
401. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
403. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
405. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-1} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
407. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-1} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
409. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d+1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
411. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d+1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
413. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d+1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
415. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3^3 x_4^{2^{d+1}-3} x_5^{2^d-2}$
417. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^3 x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
419. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^3 x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
421. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3^3 x_4^{2^d-3} x_5^{2^{d-1}-2}$
423. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
425. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
427. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
429. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
431. $x_1^{2^d-1} x_2^3 x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2^{d-1}-2}$
433. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-1} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
435. $x_1^{2^d-1} x_2^7 x_3^{2^d-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
437. $x_1^{2^d-1} x_2^7 x_3^{2^{d+1}-2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$
439. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$
441. $x_1^{2^d-1} x_2^{2^d-1} x_3^3 x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
443. $x_1^{2^d-1} x_2^{2^d-1} x_3^3 x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$
445. $x_1^{2^d-1} x_2^{2^{d+1}-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
447. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
370. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
372. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
374. $x_1^7 x_2^{2^d-5} x_3^{2^{d+1}-3} x_4^{2^{d-1}-1} x_5^{2^{d-1}-2}$
376. $x_1^7 x_2^{2^d-1} x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2^{d-1}-2}$
378. $x_1^7 x_2^{2^d-1} x_3^{2^{d+1}-2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$
380. $x_1^7 x_2^{2^d-1} x_3^{2^{d+1}-5} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
382. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^{d-1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
384. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
386. $x_1^7 x_2^{2^{d+1}-5} x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
388. $x_1^7 x_2^{2^{d+1}-1} x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
390. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d+1}-1}$
392. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d+1}-2} x_5^{2^d-1}$
394. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d+1}-2}$
396. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
398. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
400. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
402. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-3} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
404. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$
406. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
408. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d+1}-3} x_4^{2^{d-1}-2} x_5^{2^{d-1}-1}$
410. $x_1^{2^{d-1}-1} x_2^3 x_3^{2^{d+1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
412. $x_1^{2^{d-1}-1} x_2^{2^{d-1}-1} x_3^3 x_4^{2^d-3} x_5^{2^{d+1}-2}$
414. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$
416. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^3 x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
418. $x_1^{2^{d-1}-1} x_2^{2^d-1} x_3^3 x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
420. $x_1^{2^{d-1}-1} x_2^{2^{d+1}-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^d-2}$
422. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$
424. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$
426. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$
428. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$
430. $x_1^{2^d-1} x_2^3 x_3^{2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$
432. $x_1^{2^d-1} x_2^3 x_3^{2^{d+1}-2^{d-1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
434. $x_1^{2^d-1} x_2^7 x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2^{d-1}-2}$
436. $x_1^{2^d-1} x_2^7 x_3^{2^{d+1}-2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$
438. $x_1^{2^d-1} x_2^7 x_3^{2^{d+1}-5} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
440. $x_1^{2^d-1} x_2^{2^{d-1}-1} x_3^3 x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
442. $x_1^{2^d-1} x_2^{2^d-1} x_3^3 x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
444. $x_1^{2^d-1} x_2^{2^{d+1}-2^{d-1}-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^d-2}$
446. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^{d-1}-2} x_5^{2^d-1}$
448. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$

449. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^d-1} x_4^{2^d-1} x_5^{2^{d-1}-2}$ 450. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$
 451. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$ 452. $x_1^{2^{d+1}-1} x_2^3 x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$
 453. $x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^d-2}$ 454. $x_1^{2^{d+1}-1} x_2^{2^{d-1}-1} x_3^3 x_4^{2^d-3} x_5^{2^{d-1}-2}$
 455. $x_1^{2^{d+1}-1} x_2^{2^d-1} x_3^3 x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$

For $d = 4$,

456. $x_1^7 x_2^7 x_3^7 x_4^8 x_5^{31}$ 457. $x_1^7 x_2^7 x_3^7 x_4^9 x_5^{30}$ 458. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{24}$ 459. $x_1^7 x_2^7 x_3^7 x_4^{24} x_5^{15}$
 460. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^{14}$ 461. $x_1^7 x_2^7 x_3^7 x_4^{31} x_5^8$ 462. $x_1^7 x_2^7 x_3^7 x_4^{31} x_5^{31}$ 463. $x_1^7 x_2^7 x_3^7 x_4^{30} x_5^{30}$
 464. $x_1^7 x_2^7 x_3^7 x_4^{14} x_5^{23}$ 465. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{22}$ 466. $x_1^7 x_2^7 x_3^7 x_4^{22} x_5^{15}$ 467. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^{14}$
 468. $x_1^7 x_2^7 x_3^7 x_4^{30} x_5^6$ 469. $x_1^7 x_2^7 x_3^7 x_4^{31} x_5^6$ 470. $x_1^7 x_2^7 x_3^7 x_4^{24} x_5^{15}$ 471. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{22}$
 472. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{16}$ 473. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^{14}$ 474. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^8$ 475. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^6$
 476. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^{15}$ 477. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^{14}$ 478. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^7$ 479. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^6$
 480. $x_1^7 x_2^7 x_3^7 x_4^{31} x_5^8$ 481. $x_1^7 x_2^7 x_3^7 x_4^{31} x_5^6$ 482. $x_1^7 x_2^7 x_3^7 x_4^{24} x_5^{22}$ 483. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^{22}$
 484. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{16}$ 485. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^{14}$ 486. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^8$ 487. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^6$
 488. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{16}$ 489. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{16}$ 490. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^6$ 491. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^{14}$
 492. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^8$ 493. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^6$ 494. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^{16}$ 495. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^6$
 496. $x_1^7 x_2^7 x_3^7 x_4^{24} x_5^{16}$ 497. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^6$ 498. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^{16}$ 499. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^{14}$
 500. $x_1^7 x_2^7 x_3^7 x_4^{24} x_5^8$ 501. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^6$ 502. $x_1^7 x_2^7 x_3^7 x_4^{25} x_5^{16}$ 503. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^6$
 504. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^6$ 505. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^{16}$ 506. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^8$ 507. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^6$
 508. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{16}$ 509. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{16}$ 510. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^6$ 511. $x_1^7 x_2^7 x_3^7 x_4^{17} x_5^{14}$
 512. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^6$ 513. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^6$ 514. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{14}$ 515. $x_1^7 x_2^7 x_3^7 x_4^{15} x_5^{14}$
 516. $x_1^7 x_2^7 x_3^7 x_4^{13} x_5^6$ 517. $x_1^7 x_2^7 x_3^7 x_4^{13} x_5^8$ 518. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^6$ 519. $x_1^7 x_2^7 x_3^7 x_4^{23} x_5^8$
 520. $x_1^{31} x_2^7 x_3^7 x_4^8 x_5^6$

For $d \geq 5$,

456. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d-1}-3} x_4^{2^d-2} x_5^{2^{d+1}-1}$ 457. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d-1}-3} x_4^{2^d-1} x_5^{2^{d+1}-2}$
 458. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-2} x_5^{2^d-1}$ 459. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d-1}-3} x_4^{2^{d+1}-1} x_5^{2^d-2}$
 460. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d-1}-1} x_4^{2^d-3} x_5^{2^{d+1}-2}$ 461. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d-1}-1} x_4^{2^{d+1}-3} x_5^{2^d-2}$
 462. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^{d-1}-2} x_5^{2^{d+1}-1}$ 463. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^{d-1}-1} x_5^{2^{d+1}-2}$
 464. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^d-2} x_5^{2^{d+1}-2^{d-1}-1}$ 465. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^d-1} x_5^{2^{d+1}-2^{d-1}-2}$
 466. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-2} x_5^{2^d-1}$ 467. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^{d+1}-2^{d-1}-1} x_5^{2^d-2}$
 468. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^{d+1}-2} x_5^{2^{d-1}-1}$ 469. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-3} x_4^{2^{d+1}-1} x_5^{2^{d-1}-2}$
 470. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-1} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 471. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-1} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
 472. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-1} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$ 473. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^d-1} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
 474. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d+1}-3} x_4^{2^d-1} x_5^{2^d-1}$ 475. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d+1}-3} x_4^{2^{d-1}-1} x_5^{2^d-2}$
 476. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d+1}-3} x_4^{2^d-2} x_5^{2^{d-1}-1}$ 477. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d+1}-3} x_4^{2^d-1} x_5^{2^{d-1}-2}$
 478. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d+1}-1} x_4^{2^{d-1}-3} x_5^{2^d-2}$ 479. $x_1^7 x_2^{2^{d-1}-5} x_3^{2^{d+1}-1} x_4^{2^d-3} x_5^{2^{d-1}-2}$
 480. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d+1}-2}$ 481. $x_1^7 x_2^{2^{d-1}-1} x_3^{2^{d-1}-5} x_4^{2^{d+1}-3} x_5^{2^d-2}$
 482. $x_1^7 x_2^{2^d-1} x_3^{2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 483. $x_1^7 x_2^{2^d-1} x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
 484. $x_1^7 x_2^{2^d-1} x_3^{2^{d-1}-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$ 485. $x_1^7 x_2^{2^d-1} x_3^{2^{d-1}-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
 486. $x_1^7 x_2^{2^{d+1}-1} x_3^{2^{d-1}-5} x_4^{2^d-1} x_5^{2^d-2}$ 487. $x_1^7 x_2^{2^{d+1}-1} x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$
 488. $x_1^{15} x_2^{2^d-9} x_3^{2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 489. $x_1^{15} x_2^{2^d-9} x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
 490. $x_1^{15} x_2^{2^d-9} x_3^{2^{d-1}-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$ 491. $x_1^{15} x_2^{2^d-9} x_3^{2^{d-1}-5} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
 492. $x_1^{15} x_2^{2^d-9} x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2^{d-1}-2}$ 493. $x_1^{15} x_2^{2^d-9} x_3^{2^d-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^{d-1}-2}$

494. $x_1^{15} x_2^{2^d-9} x_3^{2^{d+1}-2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$ 495. $x_1^{15} x_2^{2^d-9} x_3^{2^{d+1}-2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$
 496. $x_1^{15} x_2^{2^d-9} x_3^{2^{d+1}-5} x_4^{2^{d-1}-3} x_5^{2^{d-1}-2}$ 497. $x_1^{15} x_2^{2^{d+1}-9} x_3^{2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$
 498. $x_1^{15} x_2^{2^{d+1}-9} x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$ 499. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d+1}-2}$
 500. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^{d+1}-3} x_5^{2^d-2}$ 501. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$
 502. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^d-5} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$ 503. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^d-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
 504. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^d-5} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$ 505. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^{d+1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$
 506. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^{d+1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$ 507. $x_1^{2^{d-1}-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$
 508. $x_1^{2^d-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$ 509. $x_1^{2^d-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$
 510. $x_1^{2^d-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$ 511. $x_1^{2^{d+1}-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$
 512. $x_1^{2^{d+1}-1} x_2^7 x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$

For $d = 5$,

513. $x_1^{15} x_2^{15} x_3^{15} x_4^{17} x_5^{62}$ 514. $x_1^{15} x_2^{15} x_3^{15} x_4^{49} x_5^{30}$ 515. $x_1^{15} x_2^{15} x_3^{19} x_4^{13} x_5^{62}$
 516. $x_1^{15} x_2^{15} x_3^{19} x_4^{29} x_5^{46}$ 517. $x_1^{15} x_2^{15} x_3^{19} x_4^{45} x_5^{30}$ 518. $x_1^{15} x_2^{15} x_3^{19} x_4^{61} x_5^{14}$
 519. $x_1^{15} x_2^{15} x_3^{51} x_4^{13} x_5^{30}$ 520. $x_1^{15} x_2^{15} x_3^{51} x_4^{29} x_5^{14}$

For $d \geq 6$,

513. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^{d-1}-5} x_4^{2^d-3} x_5^{2^{d+1}-2}$ 514. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^{d-1}-5} x_4^{2^{d+1}-3} x_5^{2^d-2}$
 515. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^d-5} x_4^{2^{d-1}-3} x_5^{2^{d+1}-2}$ 516. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^d-5} x_4^{2^d-3} x_5^{2^{d+1}-2^{d-1}-2}$
 517. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^d-5} x_4^{2^{d+1}-2^{d-1}-3} x_5^{2^d-2}$ 518. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^d-5} x_4^{2^{d+1}-3} x_5^{2^{d-1}-2}$
 519. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^{d+1}-5} x_4^{2^{d-1}-3} x_5^{2^d-2}$ 520. $x_1^{15} x_2^{2^{d-1}-9} x_3^{2^{d+1}-5} x_4^{2^d-3} x_5^{2^{d-1}-2}$

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